

# Orthogonal polynomials, a short introduction

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## Abstract

This paper is a short introduction to orthogonal polynomials, both the general theory and some special classes. It ends with some remarks about the usage of computer algebra for this theory. The paper will appear as a chapter in the book “Computer Algebra in Quantum Field Theory: Integration, Summation and Special Functions”, Springer-Verlag.

## 1 Introduction

This paper is a short introduction to orthogonal polynomials, both the general theory and some special classes. After the definition and first examples in Section 2, important but mainly elementary aspects of the general theory associated with the three-term recurrence relation are treated in Section 3. Sections 4, 6 and 7 discuss special classes of orthogonal polynomials, interrupted by Section 5 about Gauss quadrature. Section 8 collects some more advanced results in the general theory of orthogonal polynomials. Finally Section 9 discusses the role of computer algebra in the theory of (special) orthogonal polynomials.

Everything treated here is well-known from the literature. I mention a few books which can be recommended for more detailed study. A great classical introduction to orthogonal polynomials, both the general theory and the special polynomials, is Szegő [24]. A very readable textbook, in particular for the general theory, is Chihara [5]. As a textbook emphasizing the special theory I recommend Andrews, Askey & Roy [2]. Very good is also Ismail [10], but more focusing on the  $q$ -case. Two recent compendia of formulas for special orthogonal polynomials are Olver et al. [18, Ch. 18] and Koekoek, Lesky & Swarttouw [12, Ch. 9 and 14].

## 2 Definition of orthogonal polynomials and first examples

Let  $\mathcal{P}$  be the real vector space of all polynomials in one variable with real coefficients. Assume on  $\mathcal{P}$  a (positive definite) inner product  $\langle f, g \rangle$  ( $f, g \in \mathcal{P}$ ). Orthogonalize the sequence of monomials  $1, x, x^2, \dots$  with respect to the inner product (Gram-Schmidt). This results into the sequence  $p_0(x), p_1(x), p_2(x), \dots$  of polynomials in  $x$ . So  $p_0(x) = 1$  and, if  $p_0(x), p_1(x), \dots, p_{n-1}(x)$  are already produced and mutually orthogonal, then

$$p_n(x) := x^n - \sum_{k=0}^{n-1} \frac{\langle x^n, p_k \rangle}{\langle p_k, p_k \rangle} p_k(x).$$

Indeed,  $p_n(x)$  is a linear combination of  $1, x, x^2, \dots, x^n$ , and

$$\langle p_n, p_j \rangle = \langle x^n, p_j \rangle - \sum_{k=0}^{n-1} \frac{\langle x^n, p_k \rangle}{\langle p_k, p_k \rangle} \langle p_k, p_j \rangle = \langle x^n, p_j \rangle - \frac{\langle x^n, p_j \rangle}{\langle p_j, p_j \rangle} \langle p_j, p_j \rangle = 0 \quad (j = 0, 1, \dots, n-1).$$

Throughout we will use the constants  $h_n$  and  $k_n$  associated with the orthogonal system:

$$\langle p_n, p_n \rangle = h_n, \quad p_n(x) = k_n x^n + \text{polynomial of lower degree.} \quad (2.1)$$

The  $p_n$  are unique up to a nonzero constant real factor. We may take them, for instance, *orthonormal* ( $h_n = 1$ ; this determines  $p_n$  uniquely if also  $k_n > 0$ ) or *monic* ( $k_n = 1$ ).

In general we want

$$\langle x f, g \rangle = \langle f, x g \rangle.$$

This is true, for instance, if

$$\langle f, g \rangle := \int_a^b f(x) g(x) w(x) dx \quad \text{or} \quad \langle f, g \rangle := \sum_{j=0}^{\infty} f(x_j) g(x_j) w_j$$

for a *weight function*  $w(x) \geq 0$  or for *weights*  $w_j > 0$ , respectively. These are special cases of an inner product

$$\langle f, g \rangle := \int_{\mathbb{R}} f(x) g(x) d\mu(x), \quad (2.2)$$

where  $\mu$  is a (positive) *measure*  $\mathbb{R}$ , namely the cases  $d\mu(x) = w(x) dx$  on an interval  $I$ , and  $\mu = \sum_{j=1}^{\infty} w_j \delta_{x_j}$ , respectively.

A measure  $\mu$  on  $\mathbb{R}$  can also be thought as a *non-decreasing function*  $\tilde{\mu}$  on  $\mathbb{R}$ . Then

$$\int_{\mathbb{R}} f(x) d\mu(x) = \int_{\mathbb{R}} f(x) d\tilde{\mu}(x) = \lim_{M \rightarrow \infty} \int_{-M}^M f(x) d\tilde{\mu}(x)$$

can be considered as a *Riemann-Stieltjes integral*. The measure  $\mu$  has in  $x$  a *mass point* of mass  $c > 0$  if the non-decreasing function  $\tilde{\mu}$  has a jump  $c$  at  $x$ , i.e., if  $\lim_{\delta \downarrow 0} (\tilde{\mu}(x+\delta) - \tilde{\mu}(x-\delta)) = c > 0$ . The number of mass points is countable. More generally, the *support* of the measure  $\mu$  consists of all  $x \in \mathbb{R}$  such that  $\tilde{\mu}(x+\delta) - \tilde{\mu}(x-\delta) > 0$  for all  $\delta > 0$ . This set  $\text{supp}(\mu)$  is always a closed subset of  $\mathbb{R}$ .

In the most general case let  $\mu$  be a (positive) measure on  $\mathbb{R}$  (of infinite support, i.e., not  $\mu = \sum_{j=1}^N w_j \delta_{x_j}$ ) such that  $\int_{\mathbb{R}} |x^n| d\mu(x) < \infty$  for all  $n = 0, 1, 2, \dots$ . A system  $\{p_0, p_1, p_2, \dots\}$  obtained by orthogonalization of  $\{1, x, x^2, \dots\}$  with respect to the inner product (2.2) is called a system of *orthogonal polynomials* with respect to the orthogonality measure  $\mu$ .

Here follow some first examples of explicit orthogonal polynomials.

- *Legendre polynomials*  $P_n(x)$ , orthogonal on  $[-1, 1]$  with respect to the weight function 1. Normalized by  $P_n(1) = 1$ .

- *Hermite polynomials*  $H_n(x)$ , orthogonal on  $(-\infty, \infty)$  with respect to the weight function  $e^{-x^2}$ . Normalized by  $k_n = 2^n$ .
- *Charlier polynomials*  $c_n(x, a)$ , orthogonal on the points  $x = 0, 1, 2, \dots$  with respect to the weights  $a^x/x!$  ( $a > 0$ ). Normalized by  $c_n(0; a) = 1$ .

The  $h_n$  (see (2.1)) can be computed for these examples:

$$\begin{aligned} \frac{1}{2} \int_{-1}^1 P_m(x) P_n(x) dx &= \frac{1}{2n+1} \delta_{m,n}, & \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} H_m(x) H_n(x) e^{-x^2} dx &= 2^n n! \delta_{m,n}, \\ e^{-a} \sum_{x=0}^{\infty} c_m(x, a) c_n(x, a) \frac{a^x}{x!} &= a^{-n} n! \delta_{m,n}. \end{aligned}$$

### 3 Three-term recurrence relation and some consequences

#### 3.1 Three-term recurrence relation

The following theorem is fundamental for the general theory of orthogonal polynomials.

**Theorem 3.1.** *Orthogonal polynomials  $p_n$  satisfy*

$$\begin{aligned} xp_n(x) &= a_n p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x) \quad (n > 0), \\ xp_0(x) &= a_0 p_1(x) + b_0 p_0(x) \end{aligned} \tag{3.1}$$

with  $a_n, b_n, c_n$  real constants and  $a_n c_{n+1} > 0$ . Also  $a_n = \frac{k_n}{k_{n+1}}$ ,  $\frac{c_{n+1}}{h_{n+1}} = \frac{a_n}{h_n}$ .

Moreover (Favard theorem), if polynomials  $p_n$  of degree  $n$  ( $n = 0, 1, 2, \dots$ ) satisfy (3.1) with  $a_n, b_n, c_n$  real constants and  $a_n c_{n+1} > 0$  then there exists a (positive) measure  $\mu$  on  $\mathbb{R}$  such that the polynomials  $p_n$  are orthogonal with respect to  $\mu$ .

The proof of the first part is easy. Indeed,  $xp_n(x) = \sum_{k=0}^{n+1} \alpha_k p_k(x)$ , and if  $k \leq n-2$  then  $\langle xp_n, p_k \rangle = \langle p_n, xp_k \rangle = 0$ , hence  $\alpha_k = 0$ . Furthermore,

$$c_{n+1} = \frac{\langle xp_{n+1}, p_n \rangle}{\langle p_n, p_n \rangle} = \frac{\langle xp_n, p_{n+1} \rangle}{h_n} = \frac{\langle xp_n, p_{n+1} \rangle}{\langle p_{n+1}, p_{n+1} \rangle} \frac{h_{n+1}}{h_n} = a_n \frac{h_{n+1}}{h_n}.$$

Hence  $a_n c_{n+1} = a_n^2 h_{n+1}/h_n > 0$ . Hence  $c_{n+1}/h_{n+1} = a_n/h_n$ .

The proof of the second part is much deeper (see Cihara [5, Ch. 2]).

#### Remarks

1. For orthonormal polynomials the recurrence relation (3.1) becomes

$$\begin{aligned} xp_n(x) &= a_n p_{n+1}(x) + b_n p_n(x) + a_{n-1} p_{n-1}(x) \quad (n > 0), \\ xp_0(x) &= a_0 p_1(x) + b_0 p_0(x), \end{aligned} \tag{3.2}$$

and for monic orthogonal polynomials

$$\begin{aligned} xp_n(x) &= p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x) \quad (n > 0), \\ xp_0(x) &= p_1(x) + b_0 p_0(x), \end{aligned} \tag{3.3}$$

with  $c_n = h_n/h_{n-1} > 0$  in (3.3). If orthonormal polynomials  $p_n$  satisfy (3.2) then the corresponding monic polynomials  $k_n^{-1}p_n$  satisfy (3.3) with  $c_n = a_{n-1}^2$ .

2. If the orthogonality measure is *even* ( $\mu(E) = \mu(-E)$ ) then  $p_n(-x) = (-1)^n p_n(x)$ , hence  $b_n = 0$ , so  $xp_n(x) = a_n p_{n+1}(x) + c_n p_{n-1}(x)$ . Examples of orthogonal polynomials with even orthogonality measure are the Legendre and Hermite polynomials.
3. The recurrence relation (3.1) determines the polynomials  $p_n$  uniquely (up to a constant factor because of the choice of the constant  $p_0$ ).
4. The orthogonality measure  $\mu$  for a system of orthogonal polynomials may not be unique (up to a constant positive factor). See Example 3.2.
5. If  $\mu$  is unique then  $\mathcal{P}$  is dense in  $L^2(\mu)$ . See Shohat & Tamarkin [21, Theorem 2.14].
6. If there is an orthogonality measure  $\mu$  with bounded support then  $\mu$  is unique. See Chihara [5, Ch. 2, Theorem 5.6].

### 3.2 Moments

The *moment functional*  $M: p \mapsto \langle p, 1 \rangle: \mathcal{P} \rightarrow \mathbb{R}$  associated with an orthogonality measure  $\mu$  is already determined by the rule  $M(p_n) = \langle p_n, 1 \rangle = 0$  for  $n > 0$ . Hence  $M$  is determined (up to a constant factor) by the system of orthogonal polynomials  $p_n$ , independent of the choice of the orthogonality measure, and hence  $M$  is also determined by (3.1). The same is true for the inner product  $\langle f, g \rangle = \langle fg, 1 \rangle$  on  $\mathcal{P}$ .

The moment functional  $M$  is also determined by the *moments*  $\mu_n := \langle x^n, 1 \rangle$  ( $n = 0, 1, 2, \dots$ ). The condition  $a_n c_{n+1} > 0$  is equivalent to *positive definiteness* of the moments, stated as

$$\Delta_n := \det(\mu_{i+j})_{i,j=0}^n > 0 \quad (n = 0, 1, 2, \dots).$$

For given moments  $\mu_n$  and corresponding orthogonal polynomials  $p_n$  a positive measure  $\mu$  is an orthogonality measure for the  $p_n$  iff  $\mu$  is a *solution of the (Hamburger) moment problem*

$$\int_{\mathbb{R}} x^n d\mu(x) = \mu_n \quad (n = 0, 1, 2, \dots). \tag{3.4}$$

Uniqueness of the orthogonality measure is equivalent to uniqueness of the moment problem.

**Example 3.2** (non-unique orthogonality measure). The following goes back to Stieltjes [23, §56]. In the easily verified formula

$$\int_{-\infty}^{\infty} e^{-u^2} (1 + C \sin(2\pi u)) du = \pi^{1/2}$$

make a transformation of integration variable  $u = \log x - \frac{1}{2}(n+1)$  and take  $-1 < C < 1$ . Then

$$\pi^{-\frac{1}{2}} e^{-\frac{1}{4}} \int_0^\infty x^n (1 + C \sin(2\pi \log x)) e^{-\log^2 x} dx = e^{\frac{1}{4}n(n+2)}. \quad (3.5)$$

Thus a one-parameter family of measures yields moments which are independent of  $C$ . The corresponding orthogonal polynomials  $p_n$  are a special case of the *Stieltjes-Wigert polynomials* [12, §14.27]:  $p_n(x) = S_n(q^{\frac{1}{2}}x; q)$  with  $q = e^{-\frac{1}{2}}$ , see Christiansen [6, p.223].

It is also elementary to show that

$$\left( \sum_{k=-\infty}^{\infty} e^{-\frac{1}{4}k^2} \right)^{-1} \sum_{k=-\infty}^{\infty} e^{-\frac{1}{2}kn} e^{-\frac{1}{4}(k+1)^2} = e^{\frac{1}{4}n(n+2)}, \quad (3.6)$$

which means that the same moments, up to a constant factor, as in (3.5) are obtained with the measure  $\sum_{k=-\infty}^{\infty} e^{-\frac{1}{4}(k+1)^2} \delta_{\exp(-\frac{1}{2}k)}$ .

### 3.3 Christoffel-Darboux formula

Let  $\mathcal{P}_n$  be the space of polynomials of degree  $\leq n$ . Let  $\{p_n\}$  be a system of orthogonal polynomials with respect to the measure  $\mu$ . The *Christoffel-Darboux kernel* is defined by

$$K_n(x, y) := \sum_{j=0}^n \frac{p_j(x)p_j(y)}{h_j}. \quad (3.7)$$

Then

$$(\Pi_n f)(x) := \int_{\mathbb{R}} K_n(x, y) f(y) d\mu(y)$$

defines an orthogonal projection  $\Pi_n: \mathcal{P} \rightarrow \mathcal{P}_n$ . Indeed, if  $f(y) = \sum_{k=0}^{\infty} \alpha_k p_k(y)$  (finite sum) then

$$(\Pi_n f)(x) = \sum_{j=0}^n p_j(x) \sum_{k=0}^{\infty} \frac{\alpha_k}{h_j} \int_{\mathbb{R}} p_j(y) p_k(y) d\mu(y) = \sum_{j=0}^n \alpha_j p_j(x).$$

The *Christoffel-Darboux formula* for  $K_n(x, y)$  given by (3.7) is as follows.

$$\sum_{j=0}^n \frac{p_j(x)p_j(y)}{h_j} = \begin{cases} \frac{k_n}{h_n k_{n+1}} \frac{p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)}{x - y} & (x \neq y), \\ \frac{k_n}{h_n k_{n+1}} (p'_{n+1}(x)p_n(x) - p'_n(x)p_{n+1}(x)) & (x = y). \end{cases} \quad (3.8)$$

$$(3.9)$$

For the proof of (3.8) note that

$$\begin{aligned} xp_j(x) &= a_j p_{j+1}(x) + b_j p_j(x) + c_j p_{j-1}(x), \\ yp_j(y) &= a_j p_{j+1}(y) + b_j p_j(y) + c_j p_{j-1}(y). \end{aligned}$$

Hence

$$\frac{(x-y)p_j(x)p_j(y)}{h_j} = \frac{a_j}{h_j}(p_{j+1}(x)p_j(y) - p_j(x)p_{j+1}(y)) - \frac{c_j}{h_j}(p_j(x)p_{j-1}(y) - p_{j-1}(x)p_j(y)).$$

Use that  $c_j/h_j = a_{j-1}/h_{j-1}$ . Sum from  $j = 0$  to  $n$ . Use that  $a_n = k_n/k_{n+1}$ . This yields (3.8). For (3.9) let  $y \rightarrow x$  in (3.8).

### 3.4 Zeros of orthogonal polynomials

**Theorem 3.3.** *Let  $p_n$  be an orthogonal polynomial of degree  $n$ . Let  $\mu$  have support within the closure of the interval  $(a, b)$ . Then  $p_n$  has  $n$  distinct zeros on  $(a, b)$ . Furthermore, the zeros of  $p_n$  and  $p_{n+1}$  alternate.*

**Proof** For the proof of the first part suppose  $p_n$  has precisely  $k < n$  sign changes on  $(a, b)$  at  $x_1, x_2, \dots, x_k$ . Hence, after possibly multiplying  $p_n$  by  $-1$ , we have  $p_n(x)(x - x_1) \dots (x - x_k) \geq 0$  on  $[a, b]$ . Hence  $\int_a^b p_n(x)(x - x_1) \dots (x - x_k) d\mu(x) > 0$ . But by orthogonality we have  $\int_a^b p_n(x)(x - x_1) \dots (x - x_k) d\mu(x) = 0$ . Contradiction.  $\square$

For the proof of the second part use (3.9): If  $k_n, k_{n+1} > 0$  then

$$p'_{n+1}(x)p_n(x) - p'_n(x)p_{n+1}(x) = \frac{h_n k_{n+1}}{k_n} \sum_{j=0}^n \frac{p_j(x)^2}{h_j} > 0.$$

Hence, if  $y, z$  are two successive zeros of  $p_{n+1}$  then

$$p'_{n+1}(y)p_n(y) > 0, \quad p'_{n+1}(z)p_n(z) > 0.$$

Since  $p'_{n+1}(y)$  and  $p'_{n+1}(z)$  have opposite signs,  $p_n(y)$  and  $p_n(z)$  must have opposite signs. Hence  $p_n$  must have a zero in the interval  $(y, z)$ .  $\square$

### 3.5 Kernel polynomials

Recall the Christoffel-Darboux formula (3.8). Suppose the orthogonality measure  $\mu$  has support within  $(-\infty, b]$  and fix  $y \geq b$ . Then

$$\int_{-\infty}^b K_n(x, y) x^k (y - x) d\mu(x) = y^k (y - y) = 0 \quad (k < n).$$

Hence  $x \mapsto q_n(x) := K_n(x, y)$  is an orthogonal polynomial of degree  $n$  on  $(-\infty, b]$  with respect to the measure  $(y - x) d\mu(x)$ . Hence

$$q_n(x) - q_{n-1}(x) = \frac{p_n(y)}{h_n} p_n(x),$$

$$p_n(y)p_{n+1}(x) - p_{n+1}(y)p_n(x) = \frac{h_n k_{n+1}}{k_n} (x - y)q_n(x).$$

The orthogonal polynomials  $q_n$  are called *kernel polynomials*. Of course, they depend on the choice of  $\mu$  and of  $y$ .

## 4 Very classical orthogonal polynomials

These are the Jacobi, Laguerre and Hermite polynomials. They are usually called *classical orthogonal polynomials*, but I prefer to call them *very classical* and to consider all polynomials in the ( $q$ -)Askey scheme (see Sections 6 and 7) as classical.

We will need *hypergeometric series* [2, Ch. 2]:

$${}_rF_s \left( \begin{matrix} a_1, \dots, a_r \\ b_s, \dots, b_s \end{matrix}; z \right) := \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_r)_k}{(b_s)_k \dots (b_s)_k} \frac{z^k}{k!}, \quad (4.1)$$

where  $(a)_k := a(a+1) \dots (a+k-1)$  for  $k = 1, 2, \dots$  and  $(a)_0 := 1$  is the *shifted factorial*. If one of the upper parameters in (4.1) equals a non-positive integer  $-n$  then the series terminates after the term with  $k = n$ .

### 4.1 Jacobi polynomials

*Jacobi polynomials*  $P_n^{(\alpha, \beta)}$  [12, §9.8] are orthogonal on  $(-1, 1)$  with respect to the weight function  $w(x) := (1-x)^\alpha (1+x)^\beta$  ( $\alpha, \beta > -1$ ) and they are normalized by  $P_n^{(\alpha, \beta)}(1) = (\alpha+1)_n / n!$ . They can be expressed as terminating Gauss hypergeometric series:

$$\begin{aligned} P_n^{(\alpha, \beta)}(x) &= \frac{(\alpha+1)_n}{n!} {}_2F_1 \left( \begin{matrix} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{matrix}; \frac{1}{2}(1-x) \right) \\ &= \sum_{k=0}^n \frac{(n+\alpha+\beta+1)_k (\alpha+k+1)_{n-k}}{k! (n-k)!} \frac{(x-1)^k}{2^k}. \end{aligned} \quad (4.2)$$

They satisfy (because of the orthogonality property) the symmetry  $P_n^{(\alpha, \beta)}(-x) = (-1)^n P_n^{(\beta, \alpha)}(x)$ . Thus we conclude (much easier than by manipulation of the hypergeometric series) that

$${}_2F_1 \left( \begin{matrix} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{matrix}; z \right) = \frac{(-1)^n (\beta+1)_n}{(\alpha+1)_n} {}_2F_1 \left( \begin{matrix} -n, n+\alpha+\beta+1 \\ \beta+1 \end{matrix}; 1-z \right).$$

For  $p_n(x) := P_n^{(\alpha, \beta)}(x)$  there is the second order differential equation

$$(1-x^2)p_n''(x) + (\beta - \alpha - (\alpha + \beta + 2)x)p_n'(x) = -n(n + \alpha + \beta + 1)p_n(x). \quad (4.3)$$

This can be split up by the *shift operator relations*

$$\frac{d}{dx} P_n^{(\alpha, \beta)}(x) = \frac{1}{2}(n + \alpha + \beta + 1) P_{n-1}^{(\alpha+1, \beta+1)}(x), \quad (4.4)$$

$$\begin{aligned} (1-x^2) \frac{d}{dx} P_{n-1}^{(\alpha+1, \beta+1)}(x) + (\beta - \alpha - (\alpha + \beta + 2)x) P_{n-1}^{(\alpha+1, \beta+1)}(x) \\ = (1-x)^{-\alpha} (1+x)^{-\beta} \frac{d}{dx} \left( (1-x)^{\alpha+1} (1+x)^{\beta+1} P_{n-1}^{(\alpha+1, \beta+1)}(x) \right) = -2n P_n^{(\alpha, \beta)}(x). \end{aligned} \quad (4.5)$$

Note that the operator  $d/dx$  acting at the left-hand side of (4.4) raises the parameters and lowers the degree of the Jacobi polynomial, while the operator acting at the left-hand side of (4.5) lowers the parameters and raises the degree. Iteration of (4.5) gives the *Rodrigues formula*

$$P_n^{(\alpha, \beta)}(x) = \frac{(-1)^n}{2^n n!} (1-x)^{-\alpha} (1+x)^{-\beta} \left( \frac{d}{dx} \right)^n \left( (1-x)^{\alpha+n} (1+x)^{\beta+n} \right).$$

### Special cases

- *Gegenbauer or ultraspherical polynomials* ( $\alpha = \beta = \lambda - \frac{1}{2}$ ):  $C_n^\lambda(x) := \frac{(2\lambda)_n}{(\lambda + \frac{1}{2})_n} P_n^{(\lambda - \frac{1}{2}, \lambda - \frac{1}{2})}(x)$ .
- *Legendre polynomials* ( $\alpha = \beta = 0$ ):  $P_n(x) := P_n^{(0,0)}(x)$ .
- *Chebyshev polynomials* ( $\alpha = \beta = \pm \frac{1}{2}$ ):

$$T_n(\cos \theta) := \cos(n\theta) = \frac{n!}{(\frac{1}{2})_n} P_n^{(-\frac{1}{2}, -\frac{1}{2})}(\cos \theta), \quad U_n(\cos \theta) := \frac{\sin(n+1)\theta}{\sin \theta} = \frac{(2)_n}{(\frac{3}{2})_n} P_n^{(\frac{1}{2}, \frac{1}{2})}(\cos \theta).$$

**Quadratic transformations** Since  $P_{2n}^{(\alpha, \alpha)}(x)$  is an even polynomial of degree  $2n$  in  $x$ , it is also a polynomial  $p_n(2x^2 - 1)$  of degree  $n$  in  $2x^2 - 1$ . For  $m \neq n$  we have

$$0 = \int_0^1 p_m(2y^2 - 1) p_n(2y^2 - 1) (1 - y^2)^\alpha dy = \text{const.} \int_{-1}^1 p_m(x) p_n(x) (1 - x)^\alpha (1 + x)^{-\frac{1}{2}} dx.$$

Hence

$$\frac{P_{2n}^{(\alpha, \alpha)}(x)}{P_{2n}^{(\alpha, \alpha)}(1)} = \frac{P_n^{(\alpha, -\frac{1}{2})}(2x^2 - 1)}{P_n^{(\alpha, -\frac{1}{2})}(1)}.$$

Similarly,

$$\frac{P_{2n+1}^{(\alpha, \alpha)}(x)}{P_{2n+1}^{(\alpha, \alpha)}(1)} = \frac{x P_n^{(\alpha, \frac{1}{2})}(2x^2 - 1)}{P_n^{(\alpha, \frac{1}{2})}(1)}.$$

**Theorem 4.1.** [5, Ch. 1, §8] *Let  $\{p_n\}$  be a system of orthogonal polynomials with respect to an even weight function  $w$  on  $\mathbb{R}$ . Then there are systems  $\{q_n\}$  and  $\{r_n\}$  of orthogonal polynomials on  $[0, \infty)$  with respect to weight functions  $x \mapsto x^{-\frac{1}{2}} w(x^{\frac{1}{2}})$  and  $x \mapsto x^{\frac{1}{2}} w(x^{\frac{1}{2}})$ , respectively, such that  $p_{2n}(x) = q_n(x^2)$  and  $p_{2n+1}(x) = x r_n(x^2)$ .*

### 4.2 Electrostatic interpretation of zeros

Let  $p_n(x) := \text{const.} P_n^{(2p-1, 2q-1)}(x) = (x - x_1)(x - x_2) \dots (x - x_n)$  be monic Jacobi polynomials ( $p, q > 0$ ). By (4.3)

$$(1 - x^2) p_n''(x) + 2(q - p - (p + q)x) p_n'(x) = -n(n + 2p + 2q - 1) p_n(x).$$



Hence

$$\begin{aligned} 0 &= 1 - x_k^2 p_n''(x_k) + 2(q - p - (p + q)x_k)p_n'(x_k) \\ &= \frac{1}{2} \frac{p_n''(x_k)}{p_n'(x_k)} + \frac{p}{x_k - 1} + \frac{q}{x_k + 1} = \sum_{j, j \neq k} \frac{1}{x_k - x_j} + \frac{p}{x_k - 1} + \frac{q}{x_k + 1}. \end{aligned}$$

This can be reformulated as

$$(\nabla V)(x_1, \dots, x_n) = 0,$$

where

$$V(y_1, \dots, y_n) := - \sum_{i < j} \log(y_j - y_i) - p \sum_j \log(1 - y_j) - q \sum_j \log(1 + y_j)$$

is the logarithmic potential obtained from  $n + 2$  charges  $q, 1, \dots, 1, p$  at successive points  $-1 < y_1 < \dots < y_n < 1$ . It achieves a minimum at the zeros of  $P_n^{(2p-1, 2q-1)}$ . This result goes back to Stieltjes [22].

### 4.3 Laguerre polynomials

*Laguerre polynomials*  $L_n^\alpha$  [12, §9.12] are orthogonal on  $[0, \infty)$  with respect to the weight function  $w(x) := x^\alpha e^{-x}$  ( $\alpha > -1$ ). They are normalized by  $L_n^\alpha(0) = (\alpha + 1)_n / n!$ . They can be expressed in terms of terminating confluent hypergeometric functions by

$$L_n^\alpha(x) = \frac{(\alpha + 1)_n}{n!} {}_1F_1\left(\begin{matrix} -n \\ \alpha + 1 \end{matrix}; x\right) = \sum_{k=0}^n \frac{(\alpha + k + 1)_{n-k}}{k! (n-k)!} (-x)^k. \quad (4.6)$$

For  $p_n(x) := L_n^\alpha(x)$  there is the second order differential equation

$$x p_n''(x) + (\alpha + 1 - x) p_n'(x) = -n p_n(x).$$

This can be split up by the shift operator relations

$$\frac{d}{dx} L_n^\alpha(x) = -L_{n-1}^{\alpha+1}(x),$$

and

$$x \frac{d}{dx} L_{n-1}^{\alpha+1}(x) + (\alpha + 1 - x) L_{n-1}^{\alpha+1}(x) = x^{-\alpha} e^x \frac{d}{dx} \left( x^{\alpha+1} e^{-x} L_{n-1}^{\alpha+1}(x) \right) = n L_n^\alpha(x). \quad (4.7)$$

Iteration of (4.7) gives the Rodrigues formula

$$L_n^\alpha(x) = \frac{x^{-\alpha} e^x}{n!} \left( \frac{d}{dx} \right)^n (x^{n+\alpha} e^{-x}).$$

#### 4.4 Hermite polynomials

Hermite polynomials  $H_n$  [12, §9.15] are orthogonal with respect to the weight function  $w(x) := e^{-x^2}$  on  $\mathbb{R}$  and they are normalized by  $H_n = 2^n x^n + \dots$ . They have the explicit expression

$$H_n(x) = n! \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{(-1)^j (2x)^{n-2j}}{j! (n-2j)!}. \quad (4.8)$$

There is the second order differential equation

$$H_n''(x) - 2xH_n'(x) = -2nH_n(x).$$

This can be split up by the shift operator relations

$$H_n'(x) = 2n H_{n-1}(x), \quad H_{n-1}'(x) - 2xH_{n-1}(x) = e^{x^2} \frac{d}{dx} \left( e^{-x^2} H_{n-1}(x) \right) = -H_n(x). \quad (4.9)$$

Iteration of the last equality in (4.9) gives the Rodrigues formula

$$H_n(x) = (-1)^n e^{x^2} \left( \frac{d}{dx} \right)^n \left( e^{-x^2} \right).$$

#### 4.5 General method to derive the standard formulas

The previous formulas can be derived by the following general method. Let  $(a, b)$  be an open interval and let  $w, w_1 > 0$  be strictly positive  $C^1$ -functions on  $(a, b)$ . Let  $\{p_n\}$  and  $\{q_n\}$  be systems of monic orthogonal polynomials on  $(a, b)$  with respect to the weight function  $w$  resp.  $w_1$ . Then under suitable boundary assumptions for  $w$  and  $w_1$  we have

$$\int_a^b p_n'(x) q_{m-1}(x) w_1(x) dx = - \int_a^b p_n(x) w(x)^{-1} \frac{d}{dx} (w_1(x) q_{m-1}(x)) w(x) dx.$$

Suppose that for certain  $a_n \neq 0$ :

$$w(x)^{-1} \frac{d}{dx} (w_1(x) x^{n-1}) = -a_n x^n + \text{polynomial of degree } < n.$$

Then we easily derive a pair of first order differentiation formulas connecting  $\{p_n\}$  and  $\{q_n\}$ , an eigenvalue equation for  $p_n$  involving a second order differential operator, and a formula connecting the quadratic norms for  $p_n$  and  $q_{n-1}$ :

$$\begin{aligned} p_n'(x) &= n q_{n-1}(x), \quad w(x)^{-1} \frac{d}{dx} (w_1(x) q_{n-1}(x)) = -a_n p_n(x), \\ w(x)^{-1} \frac{d}{dx} (w_1(x) p_n'(x)) &= -n a_n p_n(x), \\ n \int_a^b q_{n-1}(x)^2 w_1(x) dx &= a_n \int_a^b p_n(x)^2 w(x) dx. \end{aligned}$$

In particular, if we work with monic Jacobi polynomials  $p_n^{(\alpha, \beta)}$ , then  $(a, b) = (-1, 1)$ ,  $w(x) = (1-x)^\alpha(1+x)^\beta$ ,  $p_n(x) = p_n^{(\alpha, \beta)}(x)$ ,  $w_1(x) = (1-x)^{\alpha+1}(1+x)^{\beta+1}$ ,  $q_m(x) = p_m^{(\alpha+1, \beta+1)}(x)$ . Then  $a_n = (n + \alpha + \beta + 1)$ . Hence

$$\frac{d}{dx} p_n^{(\alpha, \beta)}(x) = n p_{n-1}^{(\alpha+1, \beta+1)}(x), \quad (4.10)$$

$$\left( (1-x^2) \frac{d}{dx} + (\beta - \alpha - (\alpha + \beta + 2)x) \right) p_{n-1}^{(\alpha+1, \beta+1)}(x) = -(n + \alpha + \beta + 1) p_n^{(\alpha, \beta)}(x). \quad (4.11)$$

For  $x = 1$  (4.11) yields

$$p_n^{(\alpha, \beta)}(1) = \frac{2(\alpha + 1)}{n + \alpha + \beta + 1} p_{n-1}^{(\alpha+1, \beta+1)}(1).$$

Iteration gives

$$p_n^{(\alpha, \beta)}(1) = \frac{2^n (\alpha + 1)_n}{(n + \alpha + \beta + 1)_n}. \quad (4.12)$$

So for  $p_n = \text{const. } p_n^{(\alpha, \beta)} = k_n x^n + \dots$  we know  $p_n(1)/k_n$ , independent of the normalization.

The hypergeometric series representation of Jacobi polynomials is next obtained from (4.10) by Taylor expansion:

$$\begin{aligned} \frac{p_n^{(\alpha, \beta)}(x)}{p_n^{(\alpha, \beta)}(1)} &= \sum_{k=0}^n \frac{(x-1)^k}{k!} \left( \frac{d}{dx} \right)^k p_n^{(\alpha, \beta)}(x) \Big|_{x=1} \\ &= \sum_{k=0}^n \frac{(x-1)^k}{k!} \frac{n!}{(n-k)!} \frac{p_{n-k}^{(\alpha+k, \beta+k)}(1)}{p_n^{(\alpha, \beta)}(1)} = {}_2F_1 \left( \begin{matrix} -n, n + \alpha + \beta + 1 \\ \alpha + 1 \end{matrix}; \frac{1}{2}(1-x) \right). \end{aligned}$$

The quadratic norm  $h_n$  can be obtained by iteration of

$$\int_{-1}^1 p_n^{(\alpha, \beta)}(x)^2 (1-x)^\alpha (1+x)^\beta dx = \frac{n}{n + \alpha + \beta + 1} \int_{-1}^1 p_{n-1}^{(\alpha+1, \beta+1)}(x)^2 (1-x)^{\alpha+1} (1+x)^{\beta+1} dx.$$

So for  $p_n = \text{const. } p_n^{(\alpha, \beta)} = k_n x^n + \dots$  we know  $h_n/k_n^2$ , independent of the normalization.

## 4.6 Characterization theorems

Up to a constant factors and up to transformations  $x \rightarrow ax + b$  of the argument the very classical orthogonal polynomials (Jacobi, Laguerre and Hermite) are uniquely determined as orthogonal polynomials  $p_n$  satisfying any of the following three criteria. (In fact there are more ways to characterize these polynomials, see Al-Salam [1].)

- (*Bochner theorem*) The  $p_n$  are eigenfunctions of a second order differential operator.
- The polynomials  $p'_{n+1}$  are again orthogonal polynomials.

- The polynomials are orthogonal with respect to a positive  $C^\infty$  weight function  $w$  on an open interval  $I$  and there is a polynomial  $X$  such that the *Rodrigues formula* holds on  $I$ :

$$p_n(x) = \text{const. } w(x)^{-1} \left( \frac{d}{dx} \right)^n (X(x)^n w(x)).$$

#### 4.7 Limit results

The very classical orthogonal polynomials are connected to each other by limit relations. We give these limits below for the monic versions  $p_n^{(\alpha, \beta)}, \ell_n^\alpha, h_n$  of these polynomials, and on each line we give also the corresponding limit for the weight functions:

$$\alpha^{n/2} p_n^{(\alpha, \alpha)}(x/\alpha^{1/2}) \rightarrow h_n(x), \quad (1 - x^2/\alpha)^\alpha \rightarrow e^{-x^2}, \quad \alpha \rightarrow \infty, \quad (4.13)$$

$$(-\beta/2)^n p_n^{(\alpha, \beta)}(1 - 2x/\beta) \rightarrow \ell_n^\alpha(x), \quad x^\alpha (1 - x/\beta)^\beta \rightarrow x^\alpha e^{-x}, \quad \beta \rightarrow \infty, \quad (4.14)$$

$$(2\alpha)^{-n/2} \ell_n^\alpha((2\alpha)^{1/2}x + \alpha) \rightarrow h_n(x), \quad (1 + (2/\alpha)^{1/2}x)^\alpha e^{-(2\alpha)^{1/2}x} \rightarrow e^{-x^2}, \quad \alpha \rightarrow \infty. \quad (4.15)$$

The limits of the orthogonal polynomials in (4.13) and (4.14) immediately follow from (4.2), (4.6) and (4.8). For various ways to prove (4.15) see [17, section 2].

### 5 Gauss quadrature

Let be given  $n$  real points  $x_1 < x_2 < \dots < x_n$ . Put  $p_n(x) := (x - x_1) \dots (x - x_n)$ . For  $k = 1, \dots, n$  let  $l_k$  be the unique polynomial of degree  $< n$  such that  $l_k(x_j) = \delta_{k,j}$  ( $j = 1, \dots, n$ ). This polynomial, called the *Lagrange interpolation polynomial*, equals

$$l_k(x) = \frac{\prod_{j; j \neq k} (x - x_j)}{\prod_{j; j \neq k} (x_k - x_j)} = \frac{p_n(x)}{(x - x_k) p'_n(x_k)}.$$

For all polynomials  $r$  of degree  $< n$  we have

$$r(x) = \sum_{k=1}^n r(x_k) l_k(x).$$

**Theorem 5.1** (Gauss quadrature). *Let  $p_n$  be an orthogonal polynomial with respect to  $\mu$  and let the  $l_k$  be the Lagrange interpolation polynomials associated with the zeros  $x_1, \dots, x_n$  of  $p_n$ . Put*

$$\lambda_k := \int_{\mathbb{R}} l_k(x) d\mu(x).$$

*Then*

$$= \int_{\mathbb{R}} l_k(x)^2 d\mu(x) > 0$$

*and for all polynomials of degree  $\leq 2n - 1$  we have*

$$\int_{\mathbb{R}} f(x) d\mu(x) = \sum_{k=1}^n \lambda_k f(x_k). \quad (5.1)$$

**Proof** Let  $f$  be a polynomial of degree  $\leq 2n - 1$ . Then for certain polynomials  $q$  and  $r$  of degree  $\leq n - 1$  we have  $f(x) = q(x)p_n(x) + r(x)$ . Hence  $f(x_k) = r(x_k)$  and

$$\int_{\mathbb{R}} f(x) d\mu(z) = \int_{\mathbb{R}} r(x) d\mu(x) = \sum_{k=1}^n r(x_k) \int_{\mathbb{R}} l_k(x) d\mu(x) = \sum_{k=1}^n \lambda_k r(x_k) = \sum_{k=1}^n \lambda_k f(x_k).$$

Also

$$\lambda_k = \sum_{j=1}^n \lambda_j l_k(x_j)^2 = \int_{\mathbb{R}} l_k(x)^2 d\mu(x) > 0. \quad \square$$

From (5.1) we see in particular that, for  $i, j \leq n - 1$ ,

$$h_j \delta_{i,j} = \int_{\mathbb{R}} p_i(x) p_j(x) d\mu(x) = \sum_{k=1}^n \lambda_k p_i(x_k) p_j(x_k).$$

Thus the finite system  $p_0, p_1, \dots, p_{n-1}$  forms a set of orthogonal polynomials on the finite set  $\{x_1, \dots, x_n\}$  of the  $n$  zeros of  $p_n$  with respect to the weights  $\lambda_k$  and with quadratic norms  $h_j$ . All information about this system is already contained in the finite system of recurrence relations

$$xp_j(x) = a_j p_{j+1}(x) + b_j p_j(x) + c_j p_{j-1}(x) \quad (j = 0, 1, \dots, n - 1)$$

with  $a_j c_{j+1} > 0$  ( $j = 0, 1, \dots, n - 2$ ). In particular, the  $\lambda_k$  are obtained up to a constant factor by solving the system

$$\sum_{k=1}^n \lambda_k p_j(x_k) = 0 \quad (j = 1, \dots, n - 1).$$

## 6 Askey scheme

As an example of a finite system of orthogonal polynomials as described at the end of the previous section, consider orthogonal polynomials  $p_0, p_1, \dots, p_N$  on the zeros  $0, 1, \dots, N$  of the polynomial  $p_{N+1}(x) := x(x - 1) \dots (x - N)$  with respect to nice explicit weights  $w_x$  ( $x = 0, 1, \dots, N$ ) like:

- $w_x := \binom{n}{x} p^x (1 - p)^{N-x}$  ( $0 < p < 1$ ). Then the  $p_n$  are the *Krawtchouk polynomials*

$$K_n(x; p, N) := {}_2F_1 \left( \begin{matrix} -n, -x \\ -N \end{matrix}; \frac{1}{p} \right) = \sum_{k=0}^n \frac{(-n)_k (-x)_k}{(-N)_k k!} \frac{1}{p^k}.$$

- $w_x := \frac{(\alpha + 1)_x}{x!} \frac{(\beta + 1)_{N-x}}{(N - x)!}$  ( $\alpha, \beta > -1$ ). Then the  $p_n$  are the *Hahn polynomials*

$$Q_n(x; \alpha, \beta, N) := {}_3F_2 \left( \begin{matrix} -n, n + \alpha + \beta + 1, -x \\ \alpha + 1, -N \end{matrix}; 1 \right).$$

Hahn polynomials are discrete versions of Jacobi polynomials:

$$Q_n(Nx; \alpha, \beta, N) = {}_3F_2 \left( \begin{matrix} -n, n + \alpha + \beta + 1, -Nx \\ \alpha + 1, -N \end{matrix}; 1 \right) \rightarrow {}_2F_1 \left( \begin{matrix} -n, n + \alpha + \beta + 1 \\ \alpha + 1 \end{matrix}; x \right) = \text{const. } P_n^{(\alpha, \beta)}(1 - 2x)$$

and

$$N^{-1} \sum_{x \in \{0, \frac{1}{N}, \frac{2}{N}, \dots, 1\}} Q_m(Nx; \alpha, \beta, N) Q_n(Nx; \alpha, \beta, N) w_{Nx} \rightarrow \text{const. } \int_0^1 P_m^{(\alpha, \beta)}(1 - 2x) P_n^{(\alpha, \beta)}(1 - 2x) x^\alpha (1 - x)^\beta dx.$$

Jacobi and Krawtchouk polynomials are different ways of looking at the matrix elements of the irreducible representations of  $SU(2)$ , see [16]. The  $3j$  coefficients or Clebsch-Gordan coefficients for  $SU(2)$  can be expressed as Hahn polynomials, see for instance [15].

While we saw that the Jacobi, Laguerre and Hermite polynomials are eigenfunctions of a second order differential operator,

$$A(x)p_n''(x) + B(x)p_n'(x) + C(x)p_n(x) = \lambda_n p_n(x), \quad (6.1)$$

the Hahn and Krawtchouk polynomials are examples of orthogonal polynomials  $p_n$  on  $\{0, 1, \dots, N\}$  which are eigenfunctions of a second order difference operator,

$$A(x)p_n(x - 1) + B(x)p_n(x) + C(x)p_n(x + 1) = \lambda_n p_n(x). \quad (6.2)$$

If we also allow orthogonal polynomials on the infinite set  $\{0, 1, 2, \dots\}$  then *Meixner polynomials*  $M_n(x; \beta, c)$  and *Charlier polynomials*  $C_n(x; a)$  appear. Here

$$M_n(x; \beta, c) := {}_2F_1 \left( \begin{matrix} -n, -x \\ \beta \end{matrix}; 1 - \frac{1}{c} \right), \quad w_x := \frac{(\beta)_x}{x!} c^x, \\ C_n(x; a) := {}_2F_0(-n, -x; ; -a^{-1}), \quad w_x := a^x / x!.$$

If we also include orthogonal polynomials which are eigenfunctions of a second order operator as follows,

$$A(x)p_n(x + i) + B(x)p_n(x) + C(x)p_n(x - i) = \lambda_n p_n(x), \quad (6.3)$$

then we have collected all families of orthogonal polynomials which belong to the *Hahn class*.

Similarly, with an eigenvalue equation of the form

$$A(x)p_n(q(x + 1)) + B(x)p_n(q(x)) + C(x)p_n(q(x - 1)) = \lambda_n p_n(q(x)), \quad (6.4)$$

where  $q$  is a fixed polynomial of second degree, we obtain the orthogonal polynomials on a *quadratic lattice*. All orthogonal polynomials satisfying an equation of the form (6.1)–(6.4) have been classified. There are only 13 families, depending on at most four parameters, and all expressible as hypergeometric functions,  ${}_4F_3$  in the most complicated case. They can be arranged hierarchically according to limit transitions denoted by arrows. This is the famous *Askey scheme*, see for instance [17, Fig.1].

## 7 The $q$ -case

On top of the Askey-scheme is lying the  $q$ -Askey scheme [12, beginning of Ch. 14], from which there are also arrows to the Askey scheme as  $q \rightarrow 1$ . We take always  $0 < q < 1$  and let  $q \uparrow 1$  for the limit to the classical case. Some typical examples of  $q$ -analogues of classical concepts are (see Gasper & Rahman [9]):

- $q$ -number:  $[a]_q := \frac{1 - q^a}{1 - q} \rightarrow a$
- $q$ -shifted factorial:  $(a; q)_n := \prod_{k=0}^{n-1} (1 - aq^k)$  (also for  $n = \infty$ ),  $\frac{(q^a; q)_k}{(1 - q)^a} \rightarrow (a)_k$ .
- $q$ -hypergeometric series:

$${}_{s+1}\phi_s \left( \begin{matrix} a_1, \dots, a_{s+1} \\ b_1, \dots, b_s \end{matrix}; q, z \right) := \sum_{k=0}^{\infty} \frac{(a_1; q)_k \dots (a_{s+1}; q)_k}{(b_1; q)_k \dots (b_s; q)_k} \frac{z^k}{(q; q)_k},$$

$${}_{s+1}\phi_s \left( \begin{matrix} q^{a_1}, \dots, q^{a_{s+1}} \\ q^{b_1}, \dots, q^{b_s} \end{matrix}; q, z \right) \rightarrow {}_{s+1}F_s \left( \begin{matrix} a_1, \dots, a_{s+1} \\ b_1, \dots, b_s \end{matrix}; z \right).$$

- $q$ -derivative:  $(D_q f)(x) := \frac{f(x) - f(qx)}{(1 - q)x} \rightarrow f'(x)$ .
- $q$ -integral:  $\int_0^1 f(x) d_q x := (1 - q) \sum_{k=0}^{\infty} f(q^k) q^k \rightarrow \int_0^1 f(x) dx$ .

The  $q$ -case allows more symmetry which may be broken when taking limits for  $q$  to 1. In the elliptic case [9, Ch. 11] lying above the  $q$ -case there is even more symmetry.

On the highest level in the  $q$ -Askey scheme are the *Askey-Wilson polynomials* [3]. They are given by

$$p_n(\cos \theta; a, b, c, d \mid q) := \frac{(ab; q)_n (ac; q)_n (ad; q)_n}{a^n} {}_4\phi_3 \left( \begin{matrix} q^{-n}, q^{n-1}abcd, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix}; q, q \right),$$

and they are symmetric in the parameters  $a, b, c, d$ . For suitable restrictions on the parameters they are orthogonal with respect to an explicit weight function on  $(-1, 1)$ . In the special case  $a = -c = \beta^{\frac{1}{2}}$ ,  $b = -d = (q\beta)^{\frac{1}{2}}$  we get the *continuous  $q$ -ultraspherical polynomials* [12, §14.10.1]. They satisfy the orthogonality relation

$$\int_0^\pi C_m(\cos \theta; \beta \mid q) C_n(\cos \theta; \beta \mid q) \left| \frac{(e^{2i\theta}; q)_\infty}{(\beta e^{2i\theta}; q)_\infty} \right|^2 d\theta = 0 \quad (m \neq n),$$

and they have the generating function

$$\left| \frac{(\beta e^{i\theta} t; q)_\infty}{(e^{i\theta} t; q)_\infty} \right|^2 = \sum_{n=0}^{\infty} C_n(x; \beta | q) t^n.$$

For  $q \uparrow 1$  they tend to ultraspherical (or Gegenbauer) polynomials:  $C_n(x; q^\lambda | q) \rightarrow C_n^\lambda(x)$ . The Gegenbauer polynomials have the generating function

$$(1 - 2xt + t^2)^{-\lambda} = \sum_{n=0}^{\infty} C_n^\lambda(x) t^n.$$

## 8 Some deeper properties of general orthogonal polynomials

### 8.1 True interval of orthogonality

Consider a system of orthogonal polynomials  $\{p_n\}$ . Let  $p_n$  have zeros  $x_{n,1} < x_{n,2} < \dots < x_{n,n}$ . Then

$$\begin{aligned} x_{i,i} &> x_{i+1,i} > \dots > x_{n,i} \downarrow \xi_i \geq -\infty, \\ x_{j,1} &< x_{j+1,2} < \dots < x_{n,n-j+1} \uparrow \eta_j \leq \infty. \end{aligned} \tag{8.1}$$

The closure  $I$  of the interval  $(\xi_1, \eta_1)$  is called the *true interval of orthogonality* of the system  $\{p_n\}$ . It has the following properties (see [21, p.112]).

1.  $I$  is the smallest closed interval containing all zeros  $x_{n,i}$ .
2. There is an orthogonality measure  $\mu$  for the  $p_n(x)$  such that  $I$  is the smallest closed interval containing the support of  $\mu$ .
3. If  $\mu$  is any orthogonality measure for the  $p_n(x)$  and  $J$  is a closed interval containing the support of  $\mu$  then  $I \subset J$ .

### 8.2 Criteria for bounded support of orthogonality measure

Recall the three-term recurrence relation (3.3) for a system of monic orthogonal polynomials  $\{p_n\}$ . Let  $\xi_1, \eta_1$  be as in (8.1). The following theorem gives criteria for the support of an orthogonality measure in terms of the behaviour of the coefficients  $b_n, c_n$  in (3.3) as  $n \rightarrow \infty$ .

#### Theorem 8.1.

1. ([5, p.109]) If  $\{b_n\}$  is bounded and  $\{c_n\}$  is unbounded then  $(\xi_1, \eta_1) = (-\infty, \infty)$ .
2. ([5, Theorem 2.2]) If  $\{b_n\}$  and  $\{c_n\}$  are bounded then  $[\xi_1, \eta_1]$  is bounded.
3. ([5, Theorem 4.5 and p.121]) If  $b_n \rightarrow b$  and  $c_n \rightarrow c$  ( $b, c$  finite) then  $\text{supp}(\mu)$  is bounded with at most countably many points outside  $[b - 2\sqrt{c}, b + 2\sqrt{c}]$ , and  $b \pm 2\sqrt{c}$  are limit points of  $\text{supp}(\mu)$ .



**Example 8.2.** Monic Jacobi polynomials  $p_n^{(\alpha, \beta)}(x)$ :

$$b_n = \frac{\beta^2 - \alpha^2}{(2n + \alpha + \beta)(2n + \alpha + \beta + 2)} \rightarrow 0,$$

$$c_n = \frac{4n(n + \alpha)(n + \beta)(n + \alpha + \beta)}{(2n + \alpha + \beta - 1)(2n + \alpha + \beta)^2(2n + \alpha + \beta + 1)} \rightarrow \frac{1}{4}.$$

Hence  $[b - 2\sqrt{c}, b + 2\sqrt{c}] = [-1, 1]$ .

### 8.3 Criteria for uniqueness of orthogonality measure

Put

$$\rho(z) := \left( \sum_{n=0}^{\infty} |p_n(z)|^2 \right)^{-1} \quad (z \in \mathbb{C}).$$

Then  $0 \leq \rho(z) < \infty$ . Note that  $\rho(z) = 0$  iff  $\sum_{n=0}^{\infty} |p_n(z)|^2$  diverges and that  $\rho(z) > 0$  iff  $\sum_{n=0}^{\infty} |p_n(z)|^2$  converges.

**Theorem 8.3.** ([21, pp. 49–51]) *The orthogonality measure is not unique iff  $\rho(z) > 0$  for all  $z \in \mathbb{C}$ . Equivalently, the orthogonality measure is unique iff  $\rho(z) = 0$  for some  $z \in \mathbb{C}$ .*

*In the case of a unique orthogonality measure  $\mu$ , we have  $\rho(z) = 0$  for  $z \in \mathbb{C} \setminus \mathbb{R}$  and  $\rho(x) = \mu(\{x\})$  (the mass at  $x$ ) for  $x \in \mathbb{R}$ , which implies that  $\rho(x) \neq 0$  iff  $x$  is a mass point of  $\mu$ .*

*In case of non-uniqueness, for each  $x \in \mathbb{R}$  the largest possible mass of a measure  $\mu$  at  $x$  is  $\rho(x)$  and there is a measure realizing this mass at  $x$ .*

Recall the moments  $\mu_n := \langle x^n, 1 \rangle = \int_{\mathbb{R}} x^n d\mu(x)$ , which are uniquely determined (up to a constant factor) by the system  $\{p_n\}$ , and also recall the three-term recurrence relation (3.2) for a system of orthonormal polynomials  $\{p_n\}$ .

**Theorem 8.4** (Carleman). ([21, Theorem 1.10 and pp. 47, 59]) *There is a unique orthogonality measure for the  $p_n$  if one of the following two conditions is satisfied.*

$$(i) \quad \sum_{n=1}^{\infty} \mu_{2n}^{-1/(2n)} = \infty, \quad (ii) \quad \sum_{n=1}^{\infty} a_n^{-1} = \infty.$$

**Example 8.5** (Hermite).

$$\mu_{2n} = \int_{-\infty}^{\infty} x^{2n} e^{-x^2} dx = \Gamma(n + \tfrac{1}{2}) \quad \text{and} \quad \log \Gamma(n + \tfrac{1}{2}) = n \log(n + \tfrac{1}{2}) + O(n) \quad \text{as } n \rightarrow \infty,$$

so  $\mu_{2n}^{-1/(2n)} \sim (n + \frac{1}{2})^{-\frac{1}{2}}$ . Hence  $\sum_{n=1}^{\infty} \mu_{2n}^{-1/(2n)} = \infty$ , i.e., the orthogonality measure for the Hermite polynomials is unique.

**Example 8.6** (Laguerre). Monic Laguerre polynomials  $p_n$  satisfy

$$xp_n(x) = p_{n+1}(x) + (2n + \alpha + 1)p_n(x) + n(n + \alpha)p_{n-1}(x).$$

Since  $\sum_{n=0}^{\infty} \frac{1}{(n(n + \alpha))^{1/2}} = \infty$ , the orthogonality measure is unique. Also note that

$$\frac{L_n^\alpha(0)^2}{h_n} = \frac{\Gamma(n + \alpha + 1)}{\Gamma(n + 1)\Gamma(\alpha + 1)} \sim n^\alpha.$$

Since  $\sum_{n=1}^{\infty} n^\alpha = \infty$  ( $\alpha > -1$ ) we conclude once more that the orthogonality measure is unique.

**Example 8.7** (Stieltjes-Wigert). Consider the moments  $\mu_n$  given by the right-hand side of (3.5). Then

$$\sum_{n=1}^{\infty} \mu_{2n}^{-1/(2n)} = \sum_{n=1}^{\infty} e^{-\frac{1}{2}(n+1)} < \infty.$$

Since the corresponding moment problem is undetermined, the above inequality agrees with Theorem 8.4(i). Furthermore, from [12, (14.27.4)] with  $q = e^{-\frac{1}{2}}$  we see that the corresponding orthonormal polynomials  $p_n(x) = \text{const. } S_n(q^{\frac{1}{2}}x; q)$  with  $q = e^{-\frac{1}{2}}$  have  $a_{n-1}^2 = e^{2n}(1 - e^{-\frac{1}{2}n})$ , by which  $\sum_{n=1}^{\infty} a_n^{-1} < \infty$ , in agreement with Theorem 8.4(ii).

## 8.4 Orthogonal polynomials and continued fractions

Let monic orthogonal polynomials  $p_n$  be recursively defined by

$$p_0(x) = 1, \quad p_1(x) = x - b_0, \quad xp_n(x) = p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x) \quad (n \geq 1, c_n > 0).$$

Then define monic orthogonal polynomials  $p_n^{(1)}$  by

$$p_0^{(1)}(x) = 1, \quad p_1^{(1)}(x) = x - b_1, \quad xp_n^{(1)}(x) = p_{n+1}^{(1)}(x) + b_{n+1} p_n^{(1)}(x) + c_{n+1} p_{n-1}^{(1)}(x) \quad (n \geq 1).$$

They are called *first associated orthogonal polynomials* or *numerator polynomials*.

Define

$$F_1(x) := \frac{1}{x - b_0}, \quad F_2(x) := \frac{1}{x - b_0 - \frac{c_1}{x - b_1}}, \quad F_3(x) := \frac{1}{x - b_0 - \frac{c_1}{x - b_1 - \frac{c_2}{x - b_2}}},$$

and recursively obtain  $F_{n+1}(x)$  from  $F_n(x)$  by replacing  $b_{n-1}$  by  $b_{n-1} + \frac{c_n}{x - b_n}$ . This is a *continued fraction*, which can be notated as

$$F_n(z) = \frac{1}{z - b_0 - |} \frac{|c_1}{z - b_1 - |} \cdots \frac{|c_{n-2}}{z - b_{n-2} - |} \frac{|c_{n-1}}{z - b_{n-1}}.$$

**Theorem 8.8** (essentially due to Stieltjes). ([5, Ch. 3, §4])

$$F_n(z) = \frac{p_{n-1}^{(1)}(z)}{p_n(z)} \quad \text{and} \quad p_{n-1}^{(1)}(y) = \frac{1}{\mu_0} \int_{\mathbb{R}} \frac{p_n(y) - p_n(x)}{y - x} d\mu(x).$$

**Theorem 8.9** (Markov). ([5, Ch. 3, (4.8)]) *Suppose that there is a (unique) orthogonality measure  $\mu$  of bounded support for the  $p_n$ . Let  $[\xi_1, \eta_1]$  be the true interval of orthogonality. Then*

$$\lim_{n \rightarrow \infty} F_n(z) = \frac{1}{\mu_0} \int_{\xi_1}^{\eta_1} \frac{d\mu(x)}{z - x},$$

*uniformly on compact subsets of  $\mathbb{C} \setminus [\xi_1, \eta_1]$ .*

## 8.5 Measures in case of non-uniqueness

Take  $p_n$  and  $p_n^{(1)}$  orthonormal:

$$\begin{aligned} p_0(x) &= 1, \quad p_1(x) = (x - b_0)/a_0, \\ xp_n(x) &= a_n p_{n+1}(x) + b_n p_n(x) + a_{n-1} p_{n-1}(x) \quad (n \geq 1), \\ p_0^{(1)}(x) &= 1, \quad p_1^{(1)}(x) = (x - b_1)/a_1, \\ xp_n^{(1)}(x) &= a_{n+1} p_{n+1}^{(1)}(x) + b_{n+1} p_n^{(1)}(x) + a_n p_{n-1}^{(1)}(x) \quad (n \geq 1), \end{aligned}$$

where  $a_n > 0$ . Let  $\mu_0 = 1$ ,  $\mu_1, \mu_2, \dots$  be the moments for the  $p_n$ . Suppose that the orthogonality measure for the  $p_n$  is not unique. Then the possible orthogonality measures are precisely the positive measures  $\mu$  solving the moment problem (3.4). The set of these solutions is convex and weakly compact.

We will need the following entire analytic functions.

$$\begin{aligned} A(z) &:= z \sum_{n=0}^{\infty} p_n^{(1)}(0) p_n^{(1)}(z), \quad B(z) := -1 + z \sum_{n=1}^{\infty} p_{n-1}^{(1)}(0) p_n(z), \\ C(z) &:= 1 + z \sum_{n=1}^{\infty} p_n(0) p_{n-1}^{(1)}(z), \quad D(z) = z \sum_{n=0}^{\infty} p_n(0) p_n(z). \end{aligned}$$

By a *Pick function* we mean a holomorphic function  $\phi$  mapping the open upper half plane into the closed upper half plane. Let  $\mathbf{P}$  denote the set of all Pick functions. In the theorem below we will associate with a Pick function  $\phi$  a certain measure  $\mu_\phi$ . There  $\mu_t$  for  $t \in \mathbb{R}$  will mean the measure  $\mu_\phi$  with  $\phi$  the constant Pick function  $z \mapsto t$ , and  $\mu_\infty$  will mean the measure  $\mu_\phi$  with  $\phi$  the constant function  $z \mapsto \infty$  (not a Pick function).

**Theorem 8.10** (Nevanlinna, M. Riesz). ([21, Theorem 2.12]) *Suppose the moment problem (3.4) is undetermined. The identity*

$$\int_{\mathbb{R}} \frac{d\mu_\phi(t)}{t - z} = - \frac{A(z)\phi(z) - C(z)}{B(z)\phi(z) - D(z)} \quad (\Im z > 0)$$

gives a one-to-one correspondence  $\phi \rightarrow \mu_\phi$  between  $\mathbf{P} \cup \{\infty\}$  and the set of measures solving the moment problem (3.4).

Furthermore the measures  $\mu_t$  ( $t \in \mathbb{R} \cup \{\infty\}$ ) are precisely the extremal elements of the convex set, and also precisely the measures  $\mu$  solving (3.4) for which the polynomials are dense in  $L^2(\mu)$ . All measures  $\mu_t$  are discrete. The mass points of  $\mu_t$  are the zeros of the entire function  $tB - D$  (or of  $B$  if  $t = \infty$ ).

**Example 8.11** (Stieltjes-Wigert). The measure which gives in (3.6) a solution for the moment problem associated with special Stieltjes-Wigert polynomials, has a support which is almost discrete, but not completely, since 0 is a limit point of the support. Therefore (see the above theorem) this measure cannot be extremal. As observed by Christiansen at the end of [6], finding explicit extremal measures for this case seems to be completely out of reach. Since the measure in (3.6) is not extremal, the polynomials will not be dense in the corresponding  $L^2$  space. Christiansen & Koelink [7, Theorem 3.5] give an explicit orthogonal system in this  $L^2$  space which complements the orthogonal system of Stieltjes-Wigert polynomials to a complete orthogonal system.

## 9 Orthogonal polynomials in connection with computer algebra

Undoubtedly, computer algebra is nowadays a powerful tool which many mathematicians and physicists use in daily practice for their research, often using wide spectrum computer algebra programs like Mathematica or Maple, to which further specialized packages are possibly added. This is certainly also the case for research in orthogonal polynomials, in particular when it concerns special families. Jacobi, Laguerre and Hermite polynomials can be immediately called in Mathematica and Maple, while other polynomials in the ( $q$ -)Askey scheme can be defined by their ( $q$ -)hypergeometric series interpretation. Even more geneneral special orthogonal polynomials can be generated by their three-term recurrence relation.

Typical kinds of computations being done are:

1. Checking a symbolic computation on computer which was first done by hand.
2. Doing a symbolic computation first on computer and then find a hopefully elegant derivation which can be written up.
3. Doing a symbolic computation on computer and then write in the paper something like: "By using Mathematica we found ...".
4. Checking general theorems, with (hopefully correct) proofs available, for special examples by computer algebra.
5. Formulating general conjectures in interaction with output of symbolic computation for special examples.
6. Trying to find a simple evaluation of a parameter dependent expression by extrapolating from outputs for special cases of the parameters.

7. Building large collections of formulas, to be made available on the internet, which are fully derived by computer algebra, and which can be made adaptive for the user.
8. Applying full force computer algebra, often using special purpose programs, for obtaining massive output which is a priori hopeless to get by hand or to be rewritten into an elegant expression.

While item 8 is common practice in high energy physics, I have little to say about this from my own experience. Concerning item 3 there may be a danger that we become lazy, and no longer look for an elegant analytic proof when the result was already obtained by computer algebra. In particular, many formulas for terminating hypergeometric series can be derived much quicker when we recognize them as orthogonal polynomials and use some orthogonality argument.

As an example of item 1, part of the formulas in the NIST handbook [18] was indeed checked by computer algebra. Concerning item 7, it is certainly a challenge for computer algebrists how much of a formula database for special functions can be produced purely by computer algebra. Current examples are CAOP [4] (maintained by Wolfram Koepf, Kassel) and DDMF [8] (maintained by Frédéric Chyzak et al. at INRIA).

The most spectacular success of computer algebra for special functions has been the Zeilberger algorithm, now already more than 20 years old. It is treated in several books: Petkovšek et al. [19], Koepf [13], Kauers & Paule [11]. In particular, [13] contains quite a lot of examples of application of this algorithm to special orthogonal polynomials, including the discrete and the  $q$ -case.

Various applications of computer algebra to special orthogonal polynomials can be found in other chapters of the present volume.

A very desirable application of computer algebra would be to recognize from a given three-term recurrence relation with explicit, possibly still parameter dependent coefficients, whether it comes from a system of orthogonal polynomials in the ( $q$ -)Askey scheme, and if so, which system precisely. A very heuristic algorithm was implemented in the procedure Rec2ortho [20] (started by Swarttouw and maintained by the author). It is only up to the level of 2 parameters in the Askey scheme. On the other hand Koepf & Schmersau [14] give an algorithm how to go back and forth between an explicit eigenvalue equation (6.1) or (6.2) and a corresponding three-term recurrence relation with explicit coefficients.

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